RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

THIRD YEAR

B.A./B.SC. FIFTH SEMESTER (July – December) 2014

Mid-Semester Examination, September 2014

Date : 17/09/2014

MATHEMATICS (Honours)

Time : 2 pm – 4 pm

Paper : VI

Full Marks : 50

[Use a separate answer book for each group]

Group – A

Answer **any one** question :

1. a) i) Let $f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

Examine for continuity at (0,0), existence of directional derivative at (0,0) & differentiability at (0,0). [2+2+1]

ii) State and prove the converse of Euler's theorem for a function of two real variables. [1+4]

b) Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be defined by $f(v)=(f_1(v),f_2(v),\ldots,f_m(v))$ where each $f_i:\mathbb{R}^n\to\mathbb{R}$. Prove that f is continuous at v_0 in $\mathbb{R}^n$ if and only if each f'_j is continuous at v_0, $1\le j\le m$. [5]

2. a) Given that $z(x,y)$ is a twice differentiable function of x and y, transform the expression $\dfrac{\partial^2 z}{\partial x^2}+2\dfrac{\partial^2 z}{\partial x\partial y}+\dfrac{\partial^2 z}{\partial y^2}=0$, by changing the independent variables x, y to new independent variables u & v, taking u = x+y, v = x–y, w = xy – z where $w\equiv w(u,v)$. [4]

b) Let $f:S\to\mathbb{R}$ where S be an open subset of $\mathbb{R}^2$. Let $(a,b)\in S$. Assume the existence of both first order partial derivatives in some neighbourhood of (a,b) & the continuity of one of the second order mixed partial derivatives at (a,b). Prove that the other second order mixed partial derivative exists at (a,b) and both are equal. [4]

c) i) Show that the function z defined by the equation, F(x – az, y – bz) = 0 $(a,b\in\mathbb{R}\setminus\{0\})$ where F is arbitrary differentiable function of two arguments, satisfies the equation $a\dfrac{\partial z}{\partial x}+b\dfrac{\partial z}{\partial y}=1$. [3]

ii) The roots of the equation in λ: $(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$ are u, v, w. Prove that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\dfrac{-2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$. [4]

Answer **any one** question :

3. a) Show that $\vec{\nabla}\times(\vec{\nabla}\times\vec{F})=\vec{\nabla}(\vec{\nabla}\cdot\vec{F})-\nabla^2\vec{F}$ where $\vec{F}=(F_1,F_2,F_3)$ is vector field, F_i (i = 1,2,3) is a real valued differentiable function of real variables x, y, z. [5]

b) Show that $(2x\cos y+z\sin y)dx+(xz\cos y-x^2\sin y)dy+x\sin y dz$ is an exact differential. Hence, solve the differential equation $(2x\cos y+z\sin y)dx+(xz\cos y-x^2\sin y)dy+x\sin y dz=0$. [4]

c) Value of $\operatorname{div}(\vec{r}\times\vec{a})$, where $\vec{a}$ is a constant vector is—

(i) 0 (ii) –2a (iii) 2a (iv) a [1]

4. a) Find the most general differentiable function f(r) so that $f(r)\vec{r}$ is solenoidal. [3]

b) Let $\vec{A} = (y-2x)\hat{i} + (3x+2y)\hat{j}$. Compute the circulation of $\vec{A}$ about a circle C in the x–y plane with centre at the origin and radius 2, if C is traversed in the positive direction. [3]

c) If $\vec{r} = 3t\hat{i} + 3t^2\hat{j} + 2t^3\hat{k}$, find the value of $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{a^3\vec{r}}{dt^3}\right]$. [2]

d) If $\vec{v}$ is a vector having fixed direction, then value of $\vec{\nabla}\times\vec{v}$ is

(i) $\vec{0}$ (ii) along $\vec{v}$ (iii) perpendicular to $\vec{v}$ (iv) None of these [2]

<u>Group - B</u>

5. Answer **<u>any two</u>** questions :

a) Show that a uniform triangular lamina of mass M is equi-momental with three particles, each of mass $\frac{M}{12}$ placed at the angular points and a particle of mass $\frac{3M}{4}$ placed at the centre of inertia of the triangle. [7½]

b) A rod of length 2a is suspended by a string of length ℓ attached to one end. If the string and the rod revolve about the vertical with uniform angular velocity and their inclinations to the vertical be θ and ϕ respectively, then show that

$$\frac{3\ell}{a} = \frac{(4\tan\theta - 3\tan\phi)\sin\phi}{(\tan\phi - \tan\theta)\sin\theta}$$ [7½]

c) A sphere of radius a is suspended by a fine wire from a fixed point at a distance ℓ from the centre. Show that the time of a small oscillation is given by

$2\pi\sqrt{\frac{5\ell^2 + 2a^2}{5\ell g}}\left(1 + \frac{1}{4}\sin^2\frac{\alpha}{2}\right)$ where α represents the amplitude of vibration. [7½]

6. Answer **<u>any two</u>** questions :

a) Show that the velocity of a particle moving in an ellipse about a centre of force in the focus is compounded of two constant velocities, $\frac{\mu}{h}$ perpendicular to the radius vector and $\frac{\mu e}{h}$ perpendicular to the major axis. [5]

b) Prove that, for a parabolic orbit, the time taken to move from the vertex to a point distant r from the focus is $\frac{1}{3\sqrt{\mu}}(r+\ell)\sqrt{2r-\ell}$ where 2ℓ is the latus rectum. [5]

c) A particle describes an ellipse of eccentricity e about a centre of force at a focus. When the particle is at one end of a minor axis its velocity is doubled. Prove that the new path is a hyperbola of eccentricity $\sqrt{9-8e^2}$. [5]

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